Dixmier traces of nonmeasurable commutators

Fun with integrals of nonintegrable functions

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For the algebraists

Fredholm module $(\mathcal{A}, H, F)$

- $\mathcal{A}$ (unital) C^* algebra, $\pi : \mathcal{A} \to H$, $F \in \mathcal{B}(H)$
- $F = F^*$, $F^2 = \mathrm{Id}$ mod compact
- $[F, \pi(a)]$ compact

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spectral triple $(\mathcal{A}, \mathcal{H}, D : \mathcal{D} \subset \mathcal{H} \to \mathcal{H})$

- analogue for unbounded operator D
- D selfadjoint Fredholm, $(1 + D^2)^{-1}$ compact
- Lipschitz algebra $= \{a : [D, \pi(a)]$ densely defined, bounded$\}$ dense in $\mathcal{A}$

Example: $D = -i\partial_\varphi$, $F = \frac{D}{\sqrt{1+D^2}}$ on $M = S^1$

- Fredholm module $(C(S^1), L^2(S^1), F)$
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F example of pseudodifferential operator (ΨDO)
$F \sim 2\Pi_+ - \mathrm{Id}$, $\Pi_+ =$ orthogonal projection onto $\mathrm{span}\{e^{in\varphi} : n \geq 0\}$

Study geometry encoded in $(C(S^1), L^2(S^1), F)$ and $(C(S^1), L^2(S^1), D)$

Integration

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C closed, smooth curve
parametrisation $Z : S^1 \to \mathbb{R}^d$

$$\int_C f\ ds = \int_{S^1} f\ |Z'|\ d\theta \equiv \int_{S^1} f\ |dZ|$$

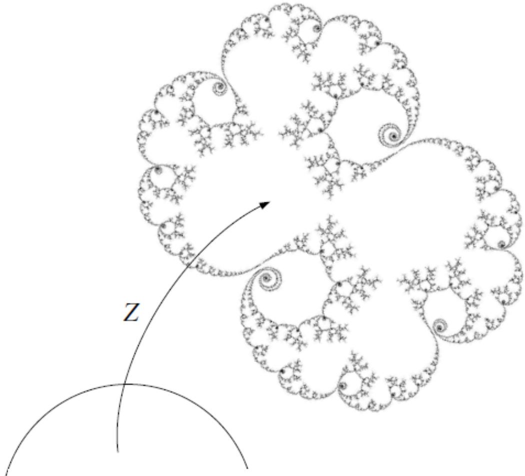
Noncommutative view on differential forms: $dZ = [D, Z]$, $[F, Z]$

$$\int_C f\ ds = \int_{S^1} f\ |dZ| \ = \mathrm{Tr}_\omega(f\ (1+D^2)^{-\frac{1}{2}}\ |[D,Z]|) \ = \mathrm{Tr}_\omega(f\ |[F,Z]|)$$

Tr_ω Dixmier trace, $\mathrm{Tr}_\omega A = \lim_{N\to\omega} \frac{1}{\log(N)} \sum_{n=1}^{N} \lambda_n(A)$ for $A \geq 0$

Integration on quasi-Fuchsian circles

Connes '94 / Sukochev et al. '16. Open: Compute normalisation λ



$$\int_C f \, d\Lambda_p = \lambda \, \mathrm{Tr}_\omega(f(Z)|[F,Z]|^p) \quad \forall f \in C_0(\mathbb{C})$$

Dixmier traces of commutators

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We need to understand the precise behaviour of $\lambda_n(A)$ as $n \to \infty$.

Related motivation:

- classical theorems in harmonic analysis:
  boundedness of commutators
- spectral theory of Hankel operators:
  ncg approach $\rightsquigarrow$ minimal regularity assumptions
- geometry/topology on nonsmooth spaces: integration,
  cohomology, mapping degrees of non-smooth functions, ...

Why commutators? – Harmonic analysis

$D^1 = \sum_{j=1}^d a_j(x)\partial_j$, $a_j \in L^\infty(\mathbb{R}^d)$, $f : \mathbb{R}^d \to \mathbb{C}$ Lipschitz $\implies$

$[D^1, f]\phi = D^1(f\phi) - fD^1\phi = \sum_j a_j(\partial_j f)\phi$ extends to bounded operator on L^p

order of commutator = (order of D^1) -1 = 0.

Calderón (1965)

D^1 ΨDO on $\mathbb{R}^d$, order 1, $f : \mathbb{R}^d \to \mathbb{C}$ Lipschitz

$\implies [D^1, f] : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ bounded and $\|[D^1, f]\|_{L^p \to L^p} \lesssim \|f\|_{Lip}$

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Coifman – Meyer (1978)

P^0 ΨDO on $\mathbb{R}^d$, order 0, $a : \mathbb{R}^d \to \mathbb{C}$ bounded mean oscillation (or C^0)

$\Longrightarrow [P^0, a] : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ bounded and $\|[P^0, a]\|_{L^p \to L^p} \lesssim \|a\|_{BMO}$

Kato – Ponce, Auscher – Taylor, ...

$[P^0, a]$ compact if e.g. $a \in C^0_c$ $\rightsquigarrow$ Spectral theory?

... related works of Pushnitski, Yafaev, also Frank (2013 –).

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An urgent challenge challenge from theoretical physics

In 1910, Hendrik Lorentz delivered lectures in Göttingen: *"Old and new problems in physics"*. An "urgent question" related to the radiation of a black body.

Prove that the density of standing electromagnetic waves inside a bounded cavity $\Omega \subset \mathbb{R}^3$ is, at high frequency, independent of the shape of Ω.

Arnold Sommerfeld had made a similar conjecture earlier in the same year, for scalar waves: Count the solutions to the Helmholtz equation with Dirichlet boundary conditions:

$$-\Delta u_k = \lambda_k u_k \text{ in } \Omega, \quad u|_{\partial\Omega} = 0 .$$

Our central objects of interest:

$$\lambda_k \sim \frac{c_3}{\mathrm{vol}(\Omega)^{2/3}} k^{2/3} + o(k^{2/3}) \text{ in 3 dimensions}$$

A young postdoc

Hermann Weyl was attending Lorentz's lectures. Within a few months he had proved the two–dimensional scalar case. Within a few years, he generalized his results to three dimensions, elastic and electromagnetic waves.

In 1913 Weyl conjectured a 2–term asymptotics in d dimensions:

$$N_\Omega(\lambda) = \#\{k : \lambda_k \leq \lambda\} \sim \frac{|S^{d-1}|\ |\Omega|}{(2\pi)^d}\lambda^{d/2} - \frac{|S^{d-2}|\ |\partial\Omega|}{4(2\pi)^{d-1}}\lambda^{(d-2)/2} + o(\lambda^{(d-2)/2})$$

Then he moved on to other areas of mathematics.

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Outline of remaining talk

general operators: $F \rightsquigarrow P^0$, $D \rightsquigarrow \mathcal{D}^1$

- A kaleidoscope of commutators – $[P^0, a]$ on $S^1 = \partial D \subset \mathbb{C}$
  - asymptotics of singular values: $\overline{\lim}_k\, k^{2\alpha} \lambda_k(A^*A) < \infty$ if $a \in C^\alpha$, $\alpha \in (0,1]$
    wide variety of non-convergence, governed by singularities
  - Dixmier traces of products: $\mathrm{Tr}_\omega\, [P^0, a]^2 = 0$ if $a \in C^{\frac{1}{2}+\varepsilon}$
    still: Connes' theorem and integral formulas

- higher dimensional generalisations and applications

Set-up

Aim: On S^1, compute $\mathrm{Tr}_\omega A$
$A =$ product of commutators $[P^0, a]$, a Hölder.

$\mathrm{Tr}_\omega A = \lim_{N\to\omega} \frac{1}{\log(N)} \sum_{n=1}^{N} \lambda_n(A)$ for $A \geq 0$
$\rightsquigarrow$ linear functional $\mathrm{Tr}_\omega : \mathcal{L}^{1,\infty} \to \mathbb{C}$ on weak-Schatten class

$$\mathcal{L}^{1,\infty} = \{A \in \mathcal{K} : \sup_n \; n\lambda_n(A^*A)^{1/2} < \infty\}$$

First question: When is $A \in \mathcal{L}^{1,\infty}$?

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Theorem

a) $A = \Pi_{n=1}^k [P_k^0, a_k] \in \mathcal{L}^{1,\infty}$ provided $a_k \in C^{\alpha_k}(S^1)$, $\sum_k \alpha_k = 1$.

b) If $\sum_k \alpha_k > 1$, then $n\lambda_n(A^*A)^{1/2} \to 0$ and hence $\mathrm{Tr}_\omega A = 0$.

This is the easy part – an upper bound on $\lambda_n(A)$.

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Pathological functionals: $C^{1/2}(S^1)^2 \ni (a,b) \to \mathrm{Tr}_\omega [P^0, a][P^0, b]$ vanishes if a or $b \in C^{1/2+\varepsilon}(S^1)$

A Lidskii theorem for Dixmier traces

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$A =$ product of commutators $[P_k^0, a_k]$, $a_k \in C^{\alpha_k}(S^1)$, $\sum_k \alpha_k = 1$.

$\mathrm{Tr}_\omega A = \lim_{N\to\omega} \frac{1}{\log(N)} \sum_{n=1}^N \lambda_n(A)$ for $A \geq 0$

Theorem

$\mathrm{Tr}_\omega A = \lim_{N\to\omega} \frac{1}{\log(2+N)} \sum_{k=0}^N \lambda_k(A)$ may be computed by Fourier series:

$$\mathrm{Tr}_\omega A = \lim_{N\to\omega} \frac{1}{\log(2N+1)} \sum_{k=-N}^{N} \langle Ae^{ikx}, e^{ikx} \rangle \ .$$

Ordering of $\sum_{k=-N}^N$ is crucial – divergent series!

Proof combines paradifferential techniques from harmonic analysis with recent advances for Dixmier traces of “modulated operators” in operator theory (Kalton, Sukochev, ...)

Examples of nice integral formulas

Proposition

Consider the Szégö projection $\Pi_+ a = \sum_{k\geq 0} a_k\, e^{ikx}$. For $a, b \in C^{1/2}(S^1)$,

$$\mathrm{Tr}_\omega\ \Pi_+[\Pi_+, a][\Pi_+, b] = -\lim_{N\to\omega} \frac{1}{\log(N+2)} \lim_{r\nearrow 1} \int_{S^1\times S^1} a_+(\bar\zeta) b_-(z) k_N(rz,\zeta)\, d\zeta\, dz\ ,$$

and

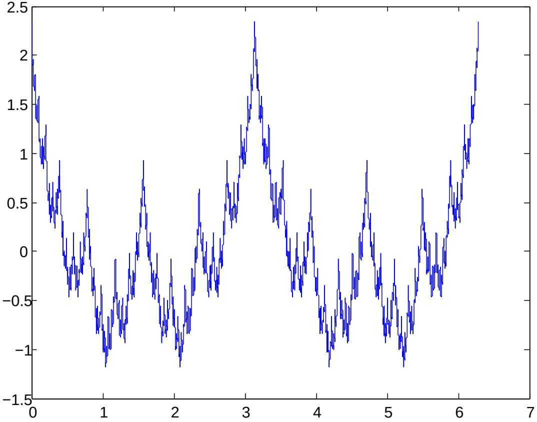
$$\mathrm{Tr}_\omega\ [\Pi_+, a][\Pi_+, b] = -\lim_{N\to\omega} \frac{1}{\log(N+2)} \lim_{r\nearrow 1} \int_{S^1\times S^1} \left\{ a_+(\bar\zeta) b_-(z) + b_+(\bar\zeta) a_-(z) \right\} k_N(rz,\zeta)\, d\zeta\, dz$$

where

$$k_N(z,\zeta) = \frac{1-(z\zeta)^{N+1}}{(1-z\zeta)^2}.$$

Recall that the right hand sides vanish if a or $b \in C^{1/2+\varepsilon}(S^1)$.

Weierstrass functions



$$W_{\alpha,b,(c_n)}(z) := \sum_{n=0}^{\infty} b^{-\alpha n} c_n (z^{b^n} + z^{-b^n}) = 2 \sum_{n=0}^{\infty} b^{-\alpha n} c_n \cos(b^n \theta), \quad \text{for} \quad z = e^{i\theta}.$$

$W_{\alpha,b,(c_n)} \in C^{\alpha}(S^1)$ if $0 < \alpha < 1$, $b \in \mathbb{N}_{>1}$, $(c_n) \in \ell^{\infty}(\mathbb{N})$.

$\lim_{N\to\infty}$ does not exist

Theorem

For $b \in \mathbb{N}_{>1}$ and $c, d \in \ell^\infty(\mathbb{N})$

$$\mathrm{Tr}_\omega \Pi_+ [\Pi_+, W_{1/2,b,(c_n)}][\Pi_+, W_{1/2,b,(d_n)}] = -\frac{1}{\log(b)} \lim_{N\to\omega} \frac{1}{N+1} \sum_{k=0}^{N} c_n\, d_n \ .$$

In particular,

$$\mathrm{Tr}_\omega [\Pi_+, W_{1/2,b,(c_n)}]^2 = -\frac{2}{\log(b)} \lim_{N\to\omega} \frac{1}{N+1} \sum_{k=0}^{N} c_n^2 \ .$$

Note that e.g. for $c_n = \sqrt{2+\cos(\log(n))}$ the sequence on the rhs $\frac{1}{N+1}\sum_{k=0}^N c_n^2 \sim 2 + \frac{1}{2}\{\sin(\log(N)) + \cos(\log(N))\}$ diverges:
Its limit depends on ω.

Any behavior in the range of the Cesaro operator $\frac{1}{N+1}\sum_{k=0}^N$ possible.

Nonmeasurable operators

Hankel measurable operators,

$$\mathfrak{hms}_{k,b} := \{c \in \ell^\infty(\mathbb{N}) : \ \mathrm{Tr}_\omega [P, W_{1/2k,b,c}]^{2k} \text{ independent of } \omega \ \}.$$

Corollary

$$\mathfrak{hms}_{1,b} = \left\{ c = (c_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}) : \ \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} c_n^2 \ \text{ exists} \right\}.$$

In particular, the inclusion $\mathfrak{hms}_{1,b} \subseteq \ell^\infty(\mathbb{N})$ is strict and does not depend on b.

Therefore we obtain countless simple examples of nonmeasurable operators. This makes integral formulas even more surprising.

Generalisations

- higher dimensional Riemannian manifolds
- non-commutative tori (see also Sukochev et al.)
- sub-Riemannian manifolds modeled on the Heisenberg group instead of $\mathbb{R}^n$

Connes' residue trace theorem

A classical computation of Dixmier traces again involves pseudodifferential operators, functions $Q = q(x, D)$ of $D = -i\partial_x$ on a closed manifold M:

$q \sim$ *smooth* rational function: $|\partial_x^\beta \partial_\xi^\alpha q(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|} \ \forall \alpha, \beta$
$m =$ order $\in \mathbb{R}$, $p =$ symbol

Assume $q \sim \sum_{j=0}^{\infty} q_{m-j}$, $q_{m-j}(x, \lambda\xi) = \lambda^{m-j} q_{m-j}(x,\xi)$ $\forall \lambda \ge 1$ $\forall |\xi| \ge 1$.

Examples

- Δ ΨDO of order $m = 2$, $q(x,\xi) = |\xi|_g^2 + \dots$
- any differential operator of order m is a ΨDO of order m, and $q(x,\xi)$ polynomial in ξ
- Δ^{-1} ΨDO of order $m = -2$, $q(x,\xi) \sim |\xi|_g^{-2} + \dots$
- Δ^α, $\alpha \in \mathbb{C}$ ΨDO of order $m = 2\mathrm{Re}\,\alpha$, $q(x,\xi) \sim |\xi|_g^{2\alpha} + \dots$
- on S^1: Szegö projection $\Pi_+ e^{inx} = e^{inx}$ if $n \ge 0$, $= 0$ else,

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Connes' residue trace theorem for Q of order $-\dim M$

$$\mathrm{Tr}_\omega Q = \frac{1}{d(2\pi)^d} \int_M \int_{|\xi|=1} q_{-\dim\ M}(x,\xi)\ d\sigma_\xi\ dx$$

In particular, independent of ω.

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Basic principle of quantization:
If $a \in C^\infty$, then $[P^0, a]$ order -1, symbol $\sim -i\{p^0, a\} = i\, \frac{\partial p^0}{\partial \xi} \frac{\partial a}{\partial x}$.

$$\mathrm{Tr}_\omega([P^0,a])^{\dim M} = \frac{1}{d(2\pi)^d} \int_M \int_{|\xi|=1} \left(i \frac{\partial p^0}{\partial \xi} \frac{\partial a}{\partial x} \right)^{\dim M} d\sigma_\xi\, dx$$

Theorem

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for $a \in Lip(M)$. In particular, independent of ω.

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for $a \in Lip(M)$. In particular, independent of ω.

For $a \in C^\alpha(M)$, $\mathrm{Tr}_\omega([P^0, a])^{\dim M}$ generally diverges.

(Implicit) Connes' theorem in general dimension

Reformulation of Connes' residue trace theorem, $d = \dim M$

$$\mathrm{Tr}_\omega\, P = \lim_{N\to\infty} \frac{1}{\log(2+N)} \sum_{k=0}^{N} \lambda_k(P)$$
$$= \frac{1}{d(2\pi)^d} \int_M \int_{|\xi|=1} p_{-d}(x,\xi)\, d\sigma_\xi\, dx$$

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$$\begin{aligned}
\mathrm{Tr}_\omega\, P &= \lim_{N\to\infty} \frac{1}{\log(2+N)} \sum_{k=0}^{N} \lambda_k(P) \\
&= \frac{1}{d(2\pi)^d} \int_M \int_{|\xi|=1} p_{-d}(x,\xi)\, d\sigma_\xi\, dx \\
&= \lim_{N\to\infty} \frac{1}{(2\pi)^d \log(2+N)} \int_M \int_{|\xi|\le N^{1/d}} p_{-d}(x,\xi)\, d\xi\, dx
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&= \lim_{N\to\infty} \frac{1}{(2\pi)^d \log(2+N)} \int_M \int_{|\xi|\le N^{1/d}} p(x,\xi)\, d\xi\, dx \\
&= \lim_{N\to\omega} \frac{1}{(2\pi)^d \log(2+N)} \int_M \int_{|\xi|\le N^{1/d}} p(x,\xi)\, d\xi\, dx
\end{aligned}
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(Implicit) Connes' theorem in general dimension

Theorem

Let A be a product of commutators $[P_k^0, a_k]$, $a_k \in C^{\alpha_k}$, $\sum_k \alpha_k = d = \dim M$.

$$\implies \quad \mathrm{Tr}_\omega\, A = \lim_{N\to\omega} \frac{1}{(2\pi)^d \log(2+N)} \int_M \int_{|\xi|\le N} p_A(x,\xi)\, d\xi\, dx\ ,$$

where p_A is the symbol of the conormal distribution $K_A(x,y)$.

More useful in terms of the integral kernel k_A:

$$\mathrm{Tr}_\omega\, A = \lim_{N\to\omega} \frac{N}{\log(N)} \int_{|x-y|<N^{-1}} k_A(x,y)\, dx\, dy\ .$$

Complex analysis: commutators as differential forms

- $\partial\Omega \subset \mathbb{C}^n$ strictly pseudoconvex, Π_+ Szegö
- $\zeta_{k,\omega} : C_{CC}^{n/k}(M)^{\otimes 2k} \to \mathbb{C}$ defined by

$$\zeta_{k,\omega}(a_1 \otimes \cdots \otimes a_{2k}) = \mathrm{Tr}_\omega \Pi_+ [[\Pi_+, a_1], [\Pi_+, a_2]] \cdots [[\Pi_+, a_{2k-1}], [\Pi_+, a_{2k}]] \, .$$

Corollaries

a) $\zeta_{k,\omega}$ defines a continuous multilinear functional $\neq 0$.

b) $k > n$: If $\exists j, \beta > \frac{n}{k}$ s.t. $a_j \in C_{CC}^{\beta}(M)$, then $\zeta_{k,\omega}(a_1 \otimes \cdots \otimes a_{2k}) = 0$.

c) $k = n$ on $Lip(M)$: Englis, Guo, Zhang (2009, 2010)

$$\zeta_{n,\omega}(a_1 \otimes \cdots \otimes a_{2n}) = \frac{1}{n!(2\pi)^n} \int_{\partial\Omega} \mathcal{L}(\overline{\partial}_b a_1, \overline{\partial}_b a_2) \cdots \mathcal{L}(\overline{\partial}_b a_{2n-1}, \overline{\partial}_b a_{2n}) \eta \wedge (d\eta)^{n-1}$$

where $\overline{\partial}_b$ boundary $\overline{\partial}$–operator, $\mathcal{L}$ dual to Levi form $\overline{\partial}\partial\varrho$ of $\partial\Omega$, η contact form.

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where $\overline{\partial}_b$ boundary $\overline{\partial}$–operator, $\mathcal{L}$ dual to Levi form $\overline{\partial}\partial\varrho$ of $\partial\Omega$, η contact form.

The map $\mathcal{R} : K_2^{alg}(C^{1/2}(S^1)) \to \mathbb{C}^*$,

$$\mathcal{R}(a, b) = \exp(\mathrm{Tr}_\omega \Pi_+ [\Pi_+, a][\Pi_+, b]) ,$$

seems to be of interest to topologists and noncommutative geometers.

References

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HG, M. Goffeng, *Commutator estimates on sub-Riemannian manifolds and applications*, preprint.